極限

例題 4　三角関数の極限／$x \to 0$ 以外の場合

(1)

$x=\dfrac{1}{t}$ とおくと，

$$\lim_{x\to\infty}\sqrt{x+3}\sin\left(\sqrt{x+2}-\sqrt{x+1}\right)=\lim_{t\to+0}\sqrt{\frac{1}{t}+3}\sin\left(\sqrt{\frac{1}{t}+2}-\sqrt{\frac{1}{t}+1}\right)$$

ここで，$\sqrt{\dfrac{1}{t}+2}-\sqrt{\dfrac{1}{t}+1}=\dfrac{\left(\dfrac{1}{t}+2\right)-\left(\dfrac{1}{t}+1\right)}{\sqrt{\dfrac{1}{t}+2}+\sqrt{\dfrac{1}{t}+1}}=\dfrac{1}{\sqrt{\dfrac{1}{t}+2}+\sqrt{\dfrac{1}{t}+1}}\xrightarrow{t\to+0}0$ だから，

$$\begin{aligned}
\lim_{x\to\infty}\sqrt{x+3}\sin\left(\sqrt{x+2}-\sqrt{x+1}\right)&=\lim_{t\to+0}\sqrt{\frac{1}{t}+3}\sin\left(\sqrt{\frac{1}{t}+2}-\sqrt{\frac{1}{t}+1}\right)\\
&=\lim_{t\to+0}\left\{\sqrt{\frac{1}{t}+3}\left(\sqrt{\frac{1}{t}+2}-\sqrt{\frac{1}{t}+1}\right)\cdot\frac{\sin\left(\sqrt{\frac{1}{t}+2}-\sqrt{\frac{1}{t}+1}\right)}{\sqrt{\frac{1}{t}+2}-\sqrt{\frac{1}{t}+1}}\right\}\\
&=\lim_{t\to+0}\left[\frac{\sqrt{\frac{1}{t}+3}\left\{\left(\frac{1}{t}+2\right)-\left(\frac{1}{t}+1\right)\right\}}{\sqrt{\frac{1}{t}+2}+\sqrt{\frac{1}{t}+1}}\cdot\frac{\sin\left(\sqrt{\frac{1}{t}+2}-\sqrt{\frac{1}{t}+1}\right)}{\sqrt{\frac{1}{t}+2}-\sqrt{\frac{1}{t}+1}}\right]\\
&=\lim_{t\to+0}\frac{\sqrt{\frac{1+3t}{t}}}{\sqrt{\frac{1+2t}{t}}+\sqrt{\frac{1+t}{t}}}\cdot\frac{\sin\left(\sqrt{\frac{1}{t}+2}-\sqrt{\frac{1}{t}+1}\right)}{\sqrt{\frac{1}{t}+2}-\sqrt{\frac{1}{t}+1}}\\
&=\lim_{t\to+0}\frac{\sqrt{1+3t}}{\sqrt{1+2t}+\sqrt{1+t}}\cdot\lim_{t\to+0}\frac{\sin\left(\sqrt{\frac{1}{t}+2}-\sqrt{\frac{1}{t}+1}\right)}{\sqrt{\frac{1}{t}+2}-\sqrt{\frac{1}{t}+1}}\\
&=\frac{1}{2}\cdot 1\\
&=\frac{1}{2}
\end{aligned}$$

例題 6　微分係数と極限

（イ）

$$\lim_{x \to a} \frac{x^p - a^p}{\sqrt{x} - \sqrt{a}} = \lim_{x \to a} \left\{ \frac{x^p - a^p}{x - a} \cdot \left(\sqrt{x} + \sqrt{a} \right) \right\}$$

ここで，$\displaystyle \lim_{x \to a} \frac{x^p - a^p}{x - a}$ について，

$f(x) = x^p$ とおくと，　$\displaystyle \lim_{x \to a} \frac{x^p - a^p}{x - a} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = f'(a)$

これと $f'(x) = px^{p-1}$ より，$\displaystyle \lim_{x \to a} \frac{x^p - a^p}{x - a} = pa^{p-1}$

よって，

$$\begin{aligned}
\lim_{x \to a} \frac{x^p - a^p}{\sqrt{x} - \sqrt{a}} &= \lim_{x \to a} \left\{ \frac{x^p - a^p}{x - a} \cdot \left(\sqrt{x} + \sqrt{a} \right) \right\} \\
&= \lim_{x \to a} \frac{x^p - a^p}{x - a} \cdot \lim_{x \to a} \left(\sqrt{x} + \sqrt{a} \right) \\
&= pa^{p-1} \cdot 2\sqrt{a} \\
&= 2pa^{p-1} \cdot a^{\frac{1}{2}} \\
&= 2pa^{p-\frac{1}{2}}
\end{aligned}$$

例題 8　数列の極限／漸化式

(2)

(i)　$n=1$ のとき $2\times\sin\dfrac{\theta}{2}\times\cos\dfrac{\theta}{2}=\sin\theta$ より

$2^n\times\sin\dfrac{\theta}{2^n}\times\cos\dfrac{\theta}{2}\times\cos\dfrac{\theta}{2^2}\times\cos\dfrac{\theta}{2^3}\times\cdots\times\cos\dfrac{\theta}{2^n}=\sin\theta$ が成り立つ。

(ii)　$n=k$ のとき $2^n\times\sin\dfrac{\theta}{2^n}\times\cos\dfrac{\theta}{2}\times\cos\dfrac{\theta}{2^2}\times\cos\dfrac{\theta}{2^3}\times\cdots\times\cos\dfrac{\theta}{2^n}=\sin\theta$ 成り立つとする。

$$
\begin{aligned}
&2^{k+1}\times\sin\frac{\theta}{2^{k+1}}\times\cos\frac{\theta}{2}\times\cos\frac{\theta}{2^2}\times\cos\frac{\theta}{2^3}\times\cdots\times\cos\frac{\theta}{2^k}\times\cos\frac{\theta}{2^{k+1}}\\
&=2\cdot 2^k\times\frac{\sin\dfrac{\theta}{2^k}}{2\cos\dfrac{\theta}{2^{k+1}}}\times\cos\frac{\theta}{2}\times\cos\frac{\theta}{2^2}\times\cos\frac{\theta}{2^3}\times\cdots\times\cos\frac{\theta}{2^k}\times\cos\frac{\theta}{2^{k+1}}\\
&=2^k\times\sin\frac{\theta}{2^k}\times\cos\frac{\theta}{2}\times\cos\frac{\theta}{2^2}\times\cos\frac{\theta}{2^3}\times\cdots\times\cos\frac{\theta}{2^k}\\
&=\sin\theta
\end{aligned}
$$

より，$n=k+1$ のときも成り立つ。

(i), (ii)より，$2^n\times\sin\dfrac{\theta}{2^n}\times\cos\dfrac{\theta}{2}\times\cos\dfrac{\theta}{2^2}\times\cos\dfrac{\theta}{2^3}\times\cdots\times\cos\dfrac{\theta}{2^n}=\sin\theta$ が成り立つことが

数学的帰納法により証明された。